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Gibbs Measures for Axiom A Flows

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We study Axiom A flows and introduce a new definition of Gibbs states which is modeled after a current one for diffeomorphisms and by which Gibbs states are locally characterized by their transformation when pulled back by conjugating homeomorphisms. We show that Gibbs states are equilibrium states and vice versa. We also show that for subshifts this equivalence can be strengthened.

KEY WORDS: Dynamical systems; hyperbolic flows; Gibbs measures; invariant measures; conjugating homeomorphisms; multipliers.

1. INTRODUCTION

For Axiom A homeomorphisms, equilibrium states are globally defined by a variational principle. It has been shown that they coincide with Gibbs states which are on the other hand defined using a local property which makes use of the hyperbolic structure induced by the hyperbolicity of the map. In this paper we extend to Axiom A flows the notion of Gibbs states using conjugating homeomorphism as it was originally introduced by Capocaccia[5] and Ruelle[9] for hyperbolic maps, and prove that they coincide with equilibrium states. Our results is in fact more general than the corresponding results for the discrete case in ref. 8. It is the flow equivalent of the quite general result in ref. 7, where it was proved that a Gibbs state in Capocaccia's original sense for a family of multipliers relates to a potential for which it is an equilibrium state.

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Section 2 we give a precise definition of conjugating maps for Axiom A flows and show that they satisfy the same properties as they do in the case of homeomorphisms. In Section 3 we introduce Gibbs states and state our main result (Theorem 5), the equivalence of Gibbs and equilibrium states. In Sections 4–6 we state and prove our main result for suspensions over subshifts of finite type (Proposition 12), where we use the fact that equilibrium states locally have a product representation. In the last section we finally use standard techniques involving Markovian sections to show that the main result holds true for general Axiom A flows. For this we show in Lemma 16 that a Gibbs state cannot be concentrated on the boundary set of a Markov partition and thus must be supported on the set of regular points, i.e., the points on which the Markov representation is unique.

2. CONJUGACY

Let M be a compact (Riemannian) manifold and Φ_t: $M \to M$ a smooth flow. A compact subset $\Omega \subset M$ which is invariant under the flow is called an Axiom A *basic set* if:

(i) For every $x \in \Omega$ the tangent space $T_x\Omega$ is the Whitney sum $E^0 \oplus E^s \oplus E^u$, where E^0 is the one-dimensional direction of the flow, and such that there exists a constant $\lambda > 0$ satisfying

$$\|D\Phi_t v\| \leqslant Ce^{-\lambda t}\,\|v\|, \qquad v \in E^s, \quad t \geqslant 0$$

$$\|D\Phi_{-t} v\| \leqslant Ce^{-\lambda t}\,\|v\|, \qquad v \in E^u, \quad t \geqslant 0$$

where we can assume that the positive constant C is equal to 1 (this is the case for an "adapted metric"). The number λ is called the *contraction parameter* of the flow.

(ii) $\Phi_t \mid \Omega$ is topologically mixing, that is, for open U, $V \subset \Omega$ the intersection $\Phi_t(U) \cap V \neq \varnothing$ for all large enough t.

(iii) $\Omega = \bigcap_{-\infty < t < \infty} \Phi_t(U)$ for some open neighborhood U of Ω.

(iv) Periodic orbits are dense in Ω.

We shall in addition also assume that the flow Φ_t is weakly mixing, i.e., that it does not factor over a simple closed orbit. If $\Omega = M$, then we say Φ_t is an Anosov flow.

Definition 1. I. Two points ξ, ζ in Ω are *conjugate* if there exist real numbers α, β such that

$$d(\Phi_{t+\alpha}(\zeta), \Phi_t(\xi)) \to 0 \quad \text{as} \quad t \to -\infty$$

$$d(\Phi_{t+\beta}(\zeta), \Phi_t(\xi)) \to 0 \quad \text{as} \quad t \to \infty$$

II. A continuous map $\varphi: \mathcal{O}_\varphi \to \Omega$, $\mathcal{O}_\varphi \subset \Omega$ open, is *conjugating* if every point $x \in \mathcal{O}_\varphi$ is conjugated to its image $\varphi(x)$ with α, β continuous functions in x, and if φ commutes with the flow Φ_t for small enough t, i.e., $\Phi_t \circ \varphi = \varphi \circ \Phi_t$, on $\mathcal{O}_\varphi \cap \Phi_{-t}(\mathcal{O}_\varphi)$. We call the two functions $\alpha(x)$ and $\beta(x)$ *delay functions* for the conjugating map φ.

Notice that the delay functions α and β are locally well defined and have the same regularity as the strong stable and unstable foliations. (Although if x lies on a periodic orbit τ, then there are many potential values of α and β all of which differ by a multiple of the minimal period of τ.) If Φ_t is the geodesic flow on a compact, negatively curved manifold, than the delay function β is the Busemann function and the conjugating points for which β is zero lie on horospheres. We can extend our notion of conjugacy to entire orbits and say that two orbits are conjugate if we can find a point on each one so that the two points are conjugate. (This is sometimes also called flow equivalence.) Obviously any two points on two conjugate orbits are conjugate.

Conjugacy is obviously an equivalence relation on Ω, with the simplest conjugating maps being given by Φ_t with $t = t(x)$ continuous in x and constant along orbits. Moreover, if φ', φ'' are two conjugating maps defined on open $\mathcal{O}_{\varphi'}$ and $\mathcal{O}_{\varphi''}$, respectively, then it is easy to see that their composition $\varphi = \varphi' \circ \varphi''$ is also a conjugating map defined on $\mathcal{O}_\varphi = \varphi'^{-1}(\mathcal{O}_{\varphi''} \cap \varphi'(\mathcal{O}_{\varphi'}))$.

The flow Φ_t is called expansive if for every $\delta > 0$ there exists a constant $\varepsilon > 0$ such that if $d(\Phi_t(\zeta), \Phi_{s(t)}(\xi)) < \varepsilon$, for all times t, for two points ζ and ξ in Ω and a continuous function $s(t)$ with $s(0) = 0$, then $\Phi_t(\xi) = \zeta$ where $|t| < \delta$.

The next two lemmas are the flow equivalents of similar results well known for hyperbolic maps.[5]

Lemma 2. Let Φ_t be expansive and $\varphi: \mathcal{O}_\varphi \to \Omega$ conjugating. Then for every $x \in \mathcal{O}_\varphi$ there exists an open $\mathcal{O}' \subset \mathcal{O}_\varphi$, $x \in \mathcal{O}'$, such that φ restricted to $\mathcal{O}'$ is a homeomorphism.

Proof. We have to show that $\varphi \mid \mathcal{O}'$ is injective for some open $\mathcal{O}' \subset \mathcal{O}_\varphi$ containing x. Let $T > 0$ be a number such that $d(\Phi_{-t}(y), \Phi_{-t+\alpha} \circ \varphi(y)) \leqslant \varepsilon/2$ and $d(\Phi_t(y), \Phi_{t+\beta} \circ \varphi(y)) \leqslant \varepsilon/2$ for $t \geqslant T$, where $\alpha = \alpha(y)$, $\beta = \beta(y)$, $y \in \mathcal{O}_\varphi$, are the delay functions of φ. Put $\alpha' = \sup_{y \in \mathcal{O}_\varphi} |\alpha(y)|$, $\beta' = \sup_{y \in \mathcal{O}_\varphi} \|\beta(y)\|$, and $\gamma = \max(\alpha, \beta)$. Let $\mathcal{O}'$ be an open neighborhood of x such that $d(\Phi_t(z), \Phi_t(x)) \leqslant \varepsilon/2$ for $|t| \leqslant T + \gamma$ and all $z \in \mathcal{O}'$. Assuming that $\varphi(y_1) = \varphi(y_2)$, for some $y_1, y_2 \in \mathcal{O}'$, we shall show that $y_1 = y_2$. We obviously have $\alpha(y_1) = \alpha(y_2)$ and $\beta(y_1) = \beta(y_2)$ and therefore:

(i) $d(\Phi_t(y_1), \Phi_t(y_2)) \leqslant \varepsilon \quad$ for $\quad |t| \leqslant T + \gamma$

(ii)
$$\begin{aligned} & d(\Phi_t(y_1), \Phi_t(y_2)) \\ & \quad \leqslant d(\Phi_t(y_1), \Phi_{t+\alpha} \circ \varphi(y_1)) + d(\Phi_{t+\alpha} \circ \varphi(y_1), \Phi_{t+\alpha} \circ \varphi(y_2)) \\ & \qquad + d(\Phi_{t+\alpha} \circ \varphi(y_2), \Phi_t(y_2)) \\ & \quad \leqslant \varepsilon/2 + \varepsilon/2 = \varepsilon \qquad \text{for} \quad t \leqslant -T \end{aligned}$$

(iii) Similarly, one shows that $d(\Phi_t(y_1), \Phi_t(y_2)) \leqslant \varepsilon$ for $t > T$.

Hence $d(\Phi_t(y_1), \Phi_t(y_2)) \leqslant \varepsilon$ for all $t \in \mathbb{R}$ and thus by expansiveness $\Phi_t(y_1) = y_2$, for some $|t| < \delta$. Since φ commutes with the flow, we get (if necessary by further shrinking the open set $\mathcal{O}'$) that $y_1 = y_2$. This proves that $y_1 = y_2$ for all $y_1, y_2 \in \mathcal{O}'$ whenever their φ images coincide. ∎

Lemma 3. Let Φ be expansive and let φ^1, φ^2 be conjugating maps defined in open $\mathcal{O}$ such that $\varphi^1(x) = \varphi^2(x)$ for some $x \in \mathcal{O}$. Then in fact $\varphi^1 \mid \mathcal{O}' = \varphi^2 \mid \mathcal{O}'$ in some neighborhood $\mathcal{O}' \subset \mathcal{O}$ of x.

Proof. Let $T > 0$ be such that $d(\Phi_{-t}(y), \Phi_{-t+\alpha} \circ \varphi^i(y)) \leqslant \varepsilon/4$, $d(\Phi_{-t}(y), \Phi_{-t+\beta} \circ \varphi^i(y)) \leqslant \varepsilon/4$ for $t > T$, $y \in \mathcal{O}$, and $i = 1, 2$. By continuity of φ^1 and φ^2 we can find an open neighborhood $\mathcal{O}' \subset \mathcal{O}$ of x such that $d(\Phi_t \circ \varphi^1(y), \Phi_t \circ \varphi^2(y)) \leqslant \varepsilon/2$ for $|t| \leqslant T + \gamma$ and $y \in \mathcal{O}'$, where γ is as in the proof of Lemma 2. By further shrinking $\mathcal{O}'$ we can also assume that $|\alpha^i(y) - \alpha^i(x)| \leqslant \varepsilon/4$, $|\beta^i(y) - \beta^i(x)| \leqslant \varepsilon/4$ for $y \in \mathcal{O}'$, $i = 1, 2$. We will show that $\varphi^1(y) = \varphi^2(y)$ for all $y \in \mathcal{O}'$. We have:

(i) $d(\Phi_t \circ \varphi^1(y), \Phi_t \circ \varphi^2(y)) \leqslant \varepsilon/2 \quad$ for $\quad |t| \leqslant T + \gamma$

(ii)
$$\begin{aligned} & d(\Phi_{-t} \circ \varphi^1(y), \Phi_{-t} \circ \varphi^2(y)) \\ & \quad \leqslant d(\Phi_{-t} \circ \varphi^1(y), \Phi_{-t+\alpha^1(y)}(y)) + d(\Phi_{-t+\alpha^1(y)}(y), \Phi_{-t+\alpha^2(y)}(y)) \\ & \qquad + d(\Phi_{-t+\alpha^1(y)}(y), \Phi_{-t} \circ \varphi^2(y)) \\ & \quad \leqslant \varepsilon/4 + \varepsilon/2 + \varepsilon/4 = \varepsilon \qquad \text{for} \quad t \geqslant T \end{aligned}$$

(iii) Similarly one shows that $d(\Phi_t \circ \varphi^1(y), \Phi_t \circ \varphi^2(y)) \leqslant \varepsilon$ for $t \geqslant T$.

Thus, since the flow is expansive, one has $\Phi_t \circ \varphi^1(y) = \varphi^2(y)$ for some $|t| < \delta$ and therefore, as φ^1 and φ^2 commute with the flow, we get $\varphi^1(y) = \varphi^2(y)$ for y in a possibly smaller $\mathcal{O}'$. ∎

Let $\varepsilon > 0$ be some number; then

$$W_\varepsilon^{ss}(x) = \{ y \in \Omega \colon d(\Phi_t(x), \Phi_t(y)) \leqslant \varepsilon \ \forall t \geqslant 0 \\ \text{and } d(\Phi_t(x), \Phi_t(y)) \to 0 \text{ as } t \to \infty \}$$

$$W_\varepsilon^{uu}(x) = \{ y \in \Omega \colon d(\Phi_{-t}(x), \Phi_{-t}(y)) \leqslant \varepsilon \ \forall t \geqslant 0 \\ \text{and } d(\Phi_{-t}(x), \Phi_{-t}(y)) \to 0 \text{ as } t \to \infty \}$$

are the local strong stable, respectively unstable, manifold through the point $x \in \Omega$. The weak stable and unstable manifolds through x are given by

$$W^s(x) = \bigcup_{-\infty < t < \infty} \Phi_t(W^{ss}(x))$$

$$W^u(x) = \bigcup_{-\infty < t < \infty} \Phi_t(W^{uu}(x))$$

where $W^{ss}(x) = \bigcup_{\varepsilon > 0} W_\varepsilon^{ss}(x)$ and $W^{uu}(x) = \bigcup_{\varepsilon > 0} W_\varepsilon^{uu}(x)$. The strong stable manifolds form a foliation transverse to the weak unstable foliation and vice versa, W^{uu} is transverse to W^s. There is a canonical local product structure on Ω, denoted by $[\cdot, \cdot]$ defined in a neighborhood of the diagonal of $\Omega \times \Omega$ and given by

$$[x, y] = W_\varepsilon^{uu}(x) \cap W_\varepsilon^s(y)$$

whenever $d(x, y) \leqslant \delta$ for some small positive δ.

If x and y are two conjugate points in Ω, then we can construct a conjugating homeomorphism φ in a neighborhood $\mathcal{O}$ of x such that $\varphi(x) = y$ as follows. Let z be close enough to x so that $d(\Phi_T(x), \Phi_T(y))$ for some $T < 0$ and put

$$z' = [\Phi_T(y), \Phi_T(z)] = W_\delta^{uu}(\Phi_T(y)) \cap W_\delta^s(\Phi_T(z))$$

Moreover, if z is close enough to x so that $d(\Phi_{T'}(z), \Phi_{T'}(y)) \leqslant \delta$ for some $T' > 0$, then we can put

$$z'' = [\Phi_{T'}(z), \Phi_{T'}(y)] = W_\delta^{uu}(\Phi_{T'}(y)) \cap W_\delta^s(\Phi_{T'}(z))$$

The image of z under φ is now defined to be

$$\varphi(z) = [\Phi_{-T'}(z''), \Phi_{-T}(z')] = W_\delta^{uu}(\Phi_{-T'}(z'')) \cap W_\delta^s(\Phi_{-T}(z'))$$

It is obvious that $\varphi(z)$ and z are conjugate and that φ is continuous in some small open neighborhood of $\mathcal{O}$ of x.

3. GIBBS STATES, EQUILIBRIUM STATES, AND RESULTS

Definition 4. I. A *family of multipliers* $\{R_\varphi$: φ conjugating$\}$ is a collection of Hölder continuous functions R_φ associated to conjugating homeomorphisms $\varphi\colon \mathcal{O}_\varphi \to \Omega$ that are flow invariant, $R_{\Phi_t \circ \varphi \circ \Phi_{-t}} \circ \Phi_t = R_\varphi$, and satisfy the multiplicative cocycle equation $R_\varphi = R_{\varphi''} \circ \varphi' \cdot R_{\varphi'}$, where $\varphi = \varphi'' \circ \varphi'$ on $\mathcal{O}_\varphi = \varphi'^{-1}(\mathcal{O}_{\varphi''} \cap \varphi'(\mathcal{O}_{\varphi'}))$ (if $\mathcal{O}_\varphi$ is nonempty).

II. A Φ_t-invariant probability measure $\hat{\mu}$ on Ω is a *Gibbs state* for a family of multipliers $\{R_\varphi\colon \varphi\}$ if for every conjugating homeomorphism $\varphi\colon \mathcal{O}_\varphi \to \Omega$ the pullback $\varphi^*\hat{\mu}$ is on $\varphi(\mathcal{O}_\varphi)$ equivalent to $\hat{\mu}$, and the Radon–Nikodym derivative satisfies

$$\frac{d\varphi^*\hat{\mu}}{d\hat{\mu}}\bigg|\, \varphi(\mathcal{O}_\varphi) = R_\varphi$$

Notice that the multipliers R_φ are constant along orbits. We also say $\hat{\mu}$ is a strong Gibbs state for a Hölder continuous function $F\colon \Omega \to \mathbb{R}$ if the multipliers are of the special form

$$R_\varphi = \exp\left\{\int_{-\infty}^{0} (F \circ \Phi_{t+\alpha} \circ \varphi - F \circ \Phi_t)\, dt + \int_\alpha^\beta [F \circ \Phi_t \circ \varphi - P(F)]\, dt + \int_0^\infty (F \circ \Phi_{t+\beta} \circ \varphi - F \circ \Phi_t)\, dt\right\} \qquad (*)$$

on $\mathcal{O}_\varphi$, with delay functions α, β, and φ conjugating. The improper integrals converge because of strong hyperbolicity of the flow and Hölder continuity of F. The number $P(F)$ is the *pressure* of F and defined is, by the variational principle,

$$P(F) = \sup_\rho \left(H(\rho) + \int F\, d\rho \right)$$

where the supremum is over all flow-invariant probability measures ρ on Ω, and the quantity $H(\rho)$ is the measure-theoretic entropy of the flow with respect to ρ. If Φ_t is weak mixing (see ref. 3), the supremum is attained by a unique probability measure which is called the *equilibrium state* for F. If F vanishes identically, we have $P(0) = H_{\mathrm{top}}(\Phi_t)$ is the topological entropy of the flow Φ_t.

Remark. Conjugacy can also be defined in the following slightly different manner. A point y in Ω is conjugate to some $x \in \Omega$ if there exists a continuous function $s\colon \mathbb{R} \to \mathbb{R}$ such that $d(\Phi_t(x), \Phi_{s(t)}(y)) \to 0$ as $|t| \to \infty$.

A probability measure is then Gibbs in the strong sense if for every Hölder continuous F on Ω the multipliers are of the form

$$R_\varphi = \exp \int_{-\infty}^{\infty} (\tilde{F} \circ \Phi_{s(t)} \circ \varphi - \tilde{F} \circ \Phi_t)\, dt$$

for every conjugating φ, where $\tilde{F} = F - P(F)$ has pressure zero.

We can now formulate our main result.

Theorem 5. Let $\Phi_t: \Omega \to \Omega$ be a weakly mixing Axiom A flow. Then:

(i) If $\hat{\mu}$ is an equilibrium state for F, Hölder continuous, then $\hat{\mu}$ is a Gibbs state in the strong sense for F.

(ii) If $\hat{\mu}$ is a strong Gibbs state for a Hölder continuous F, then it is also an equilibrium state for that same function.

Theorem 6. If $\hat{\mu}$ is a Gibbs state for some family of Hölder continuous multipliers $\{R_\varphi$: φ conjugating$\}$ with Hölder exponent $\eta > 6H_{\text{top}}(\Phi_t)/\lambda$, then $\hat{\mu}$ is the Π^* image of an equilibrium state $\bar{\mu}$ on a suspended flow $\Psi_t: \Sigma \to \Sigma$ over a subshift of finite type for a Hölder continuous function F on Σ, where $\Pi: \Sigma \to \Omega$ is one-to-one almost everywhere and satisfies $\Phi_t \circ \Pi = \Pi \circ \Psi_t$ (that is, $\hat{\mu} = \Pi^* \bar{\mu}$). The multipliers R_φ are in fact of the form given above.

The proof of this result makes heavy use of Markov partition and the fact that the flow Φ_t can be modeled by a suspended flow over a subshift of finite type. The lower bound $6H_{\text{top}}(\Phi_t)/\lambda$ for the Hölder exponent in Theorem 6 is not optimal. The optimal bound is presumably $2H_{\text{top}}(\Phi_t)/\lambda$, which we could achieve with the methods of this paper provided we used moduli of continuity[6] in defining Hölder continuous functions of shiftspaces below. However, we do not use moduli of continuity, since we need a result of ref. 7 which has not been formulated in this context.

Remarks. I. The measure m of maximal entropy is by the variational principle the equilibrium state for the function F identically vanishing. Let φ be a conjugating homeomorphism on $\mathcal{O}$ with delay functions α and β; then we get that the associated multiplier is $R_\varphi = e^{H(\alpha - \beta)}$, where $H = H_{\text{top}}(\Phi_t) = P(0)$, and $m|_{\varphi(\mathcal{O})} = \varphi^*(R_\varphi m|_{\mathcal{O}})$.

There is no measure that is invariant under conjugating homeomorphisms *per se*. This is because we have to accommodate the delay functions. This is unlike the diffeomorphism case, where in order to get convergence of forward and backward orbits there are no time delays necessary. We could define an invariant measure μ to be an equilibrium state for some function F so that $d\varphi^*\mu/d\mu = e^{(\alpha-\beta)P(F)}$. This condition is satisfied by the measure of maximal entropy, which we could call a measure "invariant" under conjugating homeomorphisms.

II. The Sinai–Bowen–Ruelle (SRB) measure: Let Ω be an attractor, that is, condition (iii) for basic sets has to be replaced by $\Omega = \bigcap_{t\geqslant 0} \Phi_t(U)$ for some open neighborhood of Ω, and assume the flow Φ_t is of class C^2. Put $\lambda_t(x)$ for the Jacobian of the linear map $D\Phi_t$ from E^u_x to $E^u_{\Phi_t(x)}$ and define

$$F(x) = -\frac{d}{dt}\log \lambda_t(x)\,|_{t=0}$$

which is a Hölder continuous function on Ω and has pressure zero.[3] The equilibrium state μ for F is called the SRB measure and is the only flow-invariant measure which is absolutely continuous on unstable leaves.

Now let φ be a conjugating homeomorphism on $\mathcal{O}\subset\Omega$ with delay functions α and β. To compute the multiplier R_φ, note that [since $P(F)=0$ and $\lambda_0=1$]

$$\int_{-\infty}^{0} (F\circ\Phi_{t+\alpha}\circ\varphi - F\circ\Phi_t)\,dt = \lim_{t\to-\infty} \log\frac{\lambda_{t+\alpha}(\varphi x)}{\lambda_\alpha(\varphi x)\,\lambda_t(x)}$$

$$\int_{\alpha}^{\beta} F\circ\Phi_t\circ dt = -\log\frac{\lambda_\beta}{\lambda_\alpha}(\varphi x)$$

$$\int_{0}^{\infty} (F\circ\Phi_{t+\beta}\circ\varphi - F\circ\Phi_t)\,dt = \lim_{t\to\infty} -\log\frac{\lambda_{t+\beta}(\varphi x)}{\lambda_\beta(\varphi x)\,\lambda_t(x)}$$

Hence we have

$$R_\varphi = \lim_{t\to\infty}\frac{\lambda_{-t+\alpha}(\varphi x)\,\lambda_t(x)}{\lambda_{-t}(x)\,\lambda_{t+\beta}(\varphi x)}$$

(the limit exists).

4. SUBSHIFTS AND DISCRETE GIBBS STATES

Let A be an $n\times n$ matrix of zeros and ones and define the *subshift* Σ as the set of points $x\in\prod_{-\infty<i<\infty}\{1, 2,..., n\}$ which satisfy the transition condition $A_{x_i,x_{i+1}}=1$ for all indices $i\in\mathbb{Z}$. On Σ we have the (two-sided) *shift transformation* σ defined by $(\sigma x)_i = x_{i+1}$ for all indices i. The topological entropy h_{top} of the shift is the largest positive eigenvalue of the matrix A. The topology on Σ is generated by the cylinder sets

$$U(x_{-n}\cdots x_n) = \{y\in\Sigma\colon y_i = x_i,\ |i|\leqslant n\}$$

where $x_{-n} \cdots x_n$ runs over all allowed finite strings in Σ of lengths $2n+1$, $n \geqslant 1$.

The *variation* of a complex function f on Σ is given by

$$\mathrm{var}_n f(z) = \sup_{z \in \Sigma} \sup\{|f(z) - f(z')| : z'_i = z_i,\ |i| \leqslant n\}$$

$n \geqslant 1$. Now let θ be a positive number; then if $\mathrm{var}_n f(x)$ decays fast enough, the quantity

$$\|f\|_\theta = \sup_{z \in \Sigma} \sup_{n \geqslant 1} e^{+2n\theta} \mathrm{var}_n f(z)$$

is finite and called the *Hölder constant* of f. Let $C_\theta(\Sigma)$ be the set of functions f which are finite with respect to the triple norm

$$|||f|||_\theta = \|f\|_\theta + \|f\|_\infty$$

where $\|\cdot\|_\infty$ is the usual supremum norm. Then $C_\theta(\Sigma)$ with the $|||\cdot|||_\theta$ norm is a Banach space.

On Σ a metric is defined by putting $d(x, y) = e^{-2\theta \cdot n(x, y)}$, $x, y \in \Sigma$, where $n(x, y)$ is the largest n such that $y_i = x_i$ for $|i| \leqslant n$. Note that in this metric f is in fact Lipshitz continuous with Lipshitz constant $\|f\|_\theta$.

Let $f: \Sigma \to \mathbb{R}$ be a Hölder continuous function on Σ; then (provided σ is mixing) the equilibrium state is the unique shift-invariant probability measure that realizes the supremum in the variational principle $p(f) = \sup_\nu (h(\nu) + \int f\, d\nu)$, where the supremum is over all shift-invariant probability measures ν on Σ. Here $h(\nu)$ is the metric entropy of σ with respect to ν. The value $p(f)$ is called the pressure of f.

Gibbs states for homeomorphisms are quite well studied and known to be equivalent to equilibrium states for a wide class of strongly hyperbolic systems. In this section we shall consider Gibbs states on the subshift Σ, which, to distinguish from the Gibbs states for Axiom A flows introduced above, we shall refer to as discrete Gibbs states. We say two points $x, y \in \Sigma$ are *conjugate* if $d(\sigma^k(x), \sigma^k(y)) \to 0$ as $|k|$ goes to infinity, and a map ψ on some (open) $\mathcal{O}_\varphi$ is *conjugating* if x and $\psi(x)$ are conjugate for all $x \in \mathcal{O}$. Note that here we do not have "delay functions" as in the case of a flow, although one could easily introduce delay functions also for the discrete action and still get the same equivalence results for discrete Gibbs and equilibrium states. For the following definition also see refs. 5 and 9.

Definition 7. I. A *family of (discrete) multipliers* on Σ is a collection of Hölder continuous functions $r_\psi: \mathcal{O}_\psi \to \Sigma$, indexed by conjugating homeomorphisms ψ, satisfying $r_{\sigma \circ \psi \circ \sigma^{-1}} \circ \sigma = r_\psi$ (shift invariance) and the multiplicative cocycle equation $r_\psi = r_{\psi''} \circ \psi' \cdot r_{\psi'}$, where $\psi = \psi'' \circ \psi'$, conjugating, is defined on $\mathcal{O}_\psi = \psi'^{-1}(\mathcal{O}_{\psi''} \cap \psi'(\mathcal{O}_{\psi'}))$, ψ', ψ'' conjugating.

II. We say a shift-invariant probability measure μ on Σ is a (*discrete*) *Gibbs state* for the family of (discrete) multipliers $\{r_\psi \in C_\theta(\Sigma)\colon \psi \text{ conjugating}\}$, $\theta > 0$, if (i) $\psi^*\mu$ is on $\psi(\mathcal{O}_\psi)$ equivalent to μ, and (ii) $d\psi^*\mu/d\mu = r_\psi$ on $\psi(\mathcal{O}_\psi)$.

To prove the second part of Theorem 5, we shall need the following result, which is proven in ref. 7, Theorem 6.

Proposition 8. Let μ be a Gibbs state on Σ for a family of multipliers $\{r_\psi \in C_\theta(\Sigma)\colon \psi \text{ conjugating in } \Sigma\}$, $\theta > h = h_{\text{top}}(\sigma)$. Then μ is the equilibrium state for some Hölder continuous function $f \in \bigcup_{0<\theta'<\theta-h} C_{\theta'/2}(\Sigma)$ and the multiplier for a conjugating ψ is in fact given by (on $\mathcal{O}_\psi$)

$$r_\psi = \exp \sum_k (f \circ \sigma^k \circ \psi - f \circ \sigma^k) \tag{$**$}$$

Conversely, an equilibrium state for some $f \in C_\theta(\Sigma)$, $\theta > 0$, is also a Gibbs state for the family of multipliers $\{r_\psi\colon \psi \text{ conjugating in } \Sigma\}$, where the multipliers r_ψ are given by the equation ($**$).

We also have the one-sided shift spaces

$$\Sigma_0 = \left\{ x \in \prod_{j \leqslant 0} \{1,..., n\}\colon A_{x_i, x_{i+1}} = 1,\ i < 0 \right\}$$

$$\Sigma_1 = \left\{ x \in \prod_{j \geqslant 1} \{1,..., n\}\colon A_{x_i, x_{i+1}} = 1,\ i \geqslant 1 \right\}$$

where the topology is, as in the two-sided case, given by one-sided cylinders. There are shift maps on these one-sided subshifts, induced by σ and σ^{-1}, which we again denote by the same symbol. However, these maps are finite to one map and only locally homeomorphism. As above we define the variation for a function f_i on Σ_i and its triple norm $|||\cdot|||_\theta$, however, with the slight difference that the Hölder constant $\|f_i\|_\theta$ is here equal to

$$\sup_{y \in \Sigma_i} \sup_{n \geqslant 1} e^{+n\theta} \operatorname{var}_n f_i(y), \qquad i = 0, 1$$

The spaces $C_\theta(\Sigma_0)$, $C_\theta(\Sigma_1)$ consist of the functions which are finite in the appropriate triple norm. Sometimes we shall use one-sided functions in a two-sided context, in which case they are understood to depend only on coordinates $\leqslant 0$, respectively on positive coordinates.

By the variational principle one has on Σ_0 and Σ_1 equilibrium states in the same way as in the two-sided case.

We say two functions f, g on Σ are *cohomologous* if there exists a function h on Σ such that $f-g=h-h\circ\sigma$. An expression of the form $f=h-h\circ\sigma$ which is cohomologous to zero is called a *coboundary*. We have the following result by Sinai which allows us to transform within a cohomology class two-sided functions into one-sided ones, however, with some loss of regularity.

Lemma 9. If $f\in C_\theta(\Sigma)$, then there exist functions f_1, $w_1\in C_{\theta/2}(\Sigma)$ such that $f_1=f+w_1-w_1\circ\sigma$ is independent of coordinates $\leqslant 0$.

In the same way on can find f_0, $w_0\in C_{\theta/2}(\Sigma)$ such that $f_0=f+w_0-w_0\circ\sigma$ only depends on coordinates $\leqslant 0$. Thus f_0 and f_1 can be considered to lie in $C_\theta(\Sigma_0)$ and $C_\theta(\Sigma_1)$, respectively.

5. SUSPENSIONS AND THE PRODUCT STRUCTURE OF EQUILIBRIUM STATES

For a strictly positive real $r\in C_\theta(\Sigma)$ one defines on the *suspension*

$$\mathbf{\Sigma}=\{(x,t)\in\Sigma\times\mathbb{R}: 0\leqslant t\leqslant r(x)\}$$

the *suspended flow* Ψ_t by $\Psi_t(x,s)=(x,t+s)$ if $0\leqslant t+s\leqslant r(x)$ and extends it to all real t by identifying the point $(x, r(x))$ with $(\sigma x, 0)$. The space $\mathbf{\Sigma}$ is metrizable, as shown in ref. 4. The system $(\mathbf{\Sigma}, \Phi_t)$ is an Axiom A flow and has (strong) stable and unstable foliations which are as in the following lemma. According to Lemma 9, let v_0, $v_1\in C_{\theta/2}(\Sigma)$ be such that $r_0=r+v_0-v_0\circ\sigma$ is a function on Σ_0 and $r_1=r+v_1-v_1\circ\sigma$ is a function on Σ_1. Then we have the following result.

Lemma 10.[6] The strong stable direction $W^{\mathrm{ss}}(xy,s)$ and the strong unstable direction $W^{\mathrm{uu}}(xy,s)$ through the point $(xy,s)\in\mathbf{\Sigma}$ are locally given by

$$\{(x'y, s+v_1(xy)-v_1(x'y)):$$
$$\qquad x'\in\Sigma_0 \text{ close to } x \text{ so that } 0\leqslant s+v_1(xy)-v_1(x'y)\leqslant r(x'y)\}$$
$$\{(xy', s-v_0(xy)+v_0(xy')):$$
$$\qquad y'\in\Sigma_1 \text{ close to } y \text{ so that } 0\leqslant s+v_0(xy)-v_0(xy')\leqslant r(xy')\}$$

respectively, and extended using the identification $(x'y, r(x'y))=(\sigma(x'y), 0)$.

Put $W^{\mathrm{u}}_\varepsilon(x)=\bigcup_{|t|\leqslant\varepsilon} W^{\mathrm{u}}_\varepsilon(\Psi_t(x))$; then, if $d(x,x')\leqslant\delta$, $0<\delta<\varepsilon$ small enough, there is a bijection $\rho_{x,x'}$ from $W^{\mathrm{u}}_\varepsilon(x)$ to $W^{\mathrm{u}}_\varepsilon(x')$ ($\varepsilon'<\varepsilon$ small enough) by shifting along the strong unstable foliation. The map $\rho_{x,x'}$: $W^{\mathrm{u}}_{\varepsilon'}(x) \to W^{\mathrm{u}}_{\varepsilon}(x')$ given by

$$\rho_{x,x'}(z) = W^{\mathrm{ss}}_{\varepsilon}(z) \cap W^{\mathrm{u}}_{\varepsilon}(x')$$

$z \in W^{\mathrm{u}}_{\varepsilon'}(x)$, locally generates Borel isomorphisms between local weak unstable leaves. Similarly, if we put $W^{\mathrm{s}}_{\varepsilon}(x) = \bigcup_{|t| \leqslant \varepsilon} W^{\mathrm{s}}_{\varepsilon}(\Psi_t(x))$ for the local unstable leaf through the point $x \in \Sigma$, then the map

$$\rho'_{x,x'}(z) = W^{\mathrm{uu}}_{\varepsilon}(z) \cap W^{\mathrm{s}}_{\varepsilon}(x')$$

defines a (local) bijection from some $W^{\mathrm{s}}_{\varepsilon'}(x)$ $(\varepsilon' < \varepsilon)$ to $W^{\mathrm{s}}_{\varepsilon}(x')$ by sliding along the strong unstable foliation.

We say a family of measures $\{\mu^{\mathrm{u}}_x : x\}$ with μ^{u}_x supported on the local weak unstable foliation through x is transverse to the strong stable foliation if the pullback $\rho^*_{x,x'}\mu^{\mathrm{u}}_{x'}$ is a measure on $W^{\mathrm{u}}_{\varepsilon}(x)$ equivalent to μ^{u}_{x}, where $\rho_{x,x'}$ is a Borel isomorphism from $W^{\mathrm{u}}_{\varepsilon}(x)$ to $W^{\mathrm{u}}_{\varepsilon'}(x')$ (for suitable $\varepsilon, \varepsilon'$) such that $\rho_{x,x'}(y) \in W^{\mathrm{ss}}(y)$, $y \in W^{\mathrm{u}}_{\varepsilon}(x)$. Similarly, one defines measures transverse to the strong unstable foliation.

To prove that equilibrium states are in fact Gibbs, we shall need the following result from ref. 6. It shows that equilibrium states are canonical products of measures on the strong stable and unstable foliations (with Lebesgue measure along the flowlines) that satisfy Margulis-type cocyle equations.

Proposition 11. Let $\bar{\mu}$ be the equilibrium state for some Hölder continuous function F on Σ. Then $\bar{\mu}$ is (up to a normalizing factor) locally given by the product $\mu^{\mathrm{ss}} \times \mu^{\mathrm{uu}} \times l$, where μ^{ss}, μ^{uu} (l is Lebesgue measure) are measures on the strong stable, respectively unstable, leaves and have the following properties:

(i) $\Psi^*_t \mu^{\mathrm{uu}} = e^{\tau_t}\mu^{\mathrm{uu}}$, $\Psi^*_t \mu^{\mathrm{ss}} = e^{-\tau_t}\mu^{\mathrm{ss}}$, where $\tau_t = \int_0^t (F \circ \Psi_s - P(F))\, ds$.

(ii) We have

$$d\mu^{\mathrm{uu}}(y) = e^{\omega_{x,y}}\, d\rho^*_{x,y}\, \mu^{\mathrm{uu}}(x), \qquad d\mu^{\mathrm{ss}}(y) = e^{\omega'_{x,y}}\, d\rho'^*_{x,y}\, \mu^{\mathrm{ss}}(x)$$

where $\rho_{x,y}$ is the map sliding along the strong unstable foliation and $\omega_{x,y} = \int_0^\infty (F \circ \Psi_s \circ \rho_{x,y} - F \circ \Psi_s)\, ds$, and $\rho'_{x,y}$ is the map sliding along the strong unstable foliation and $\omega'_{x,y} = \int_{-\infty}^0 (F \circ \Psi_s \circ \rho'_{x,y} - F \circ \Psi_s)\, ds$.

6. PROOF OF THEOREMS 5 AND 6 FOR SUSPENSIONS OVER SUBSHIFTS OF FINITE TYPE

In this section we shall prove Theorem 5 for the case when Ω is a suspension over a subshift of finite type and the flow Φ_t is the associated suspended flow. We shall make use of the fact that an equilibrium state $\bar{\mu}$ according to Proposition 12 below splits into measures on the foliation that satisfy smooth cocyle equations. We shall use these equations to separately compare the "unstable components" and the "stable components" of $\bar{\mu}$ and $\varphi^*\bar{\mu}$, where φ is any conjugating homeomorphism.

Let $\mathbf{\Sigma}$ be as before the suspension of a Hölder continuous positive function r over the subshift Σ. We call a function $F: \mathbf{\Sigma} \to \mathbb{R}$ Hölder continuous if $F(z, t_0)$ is (locally) Hölder continuous as a function in z for fixed t_0, and $F \circ \Phi_t(z, 0)$ is Hölder continuous for every $z \in \Sigma$ as a function of t. This implies in particular that $f(z) = \int_0^{r(z)} F(z, t)\, dt$ is Hölder continuous. We now formulate our main result (Theorem 5) for suspended flows.

Proposition 12. (i) If $\bar{\mu}$ is an equilibrium state for some Hölder continuous function $F: \mathbf{\Sigma} \to \mathbb{R}$, then $\bar{\mu}$ is Gibbs in the strong sense.

(ii) If $\bar{\mu}$ is strongly Gibbs for a Hölder continuous F, then it is also an equilibrium state for that same function.

We shall first prove part (i) of Proposition 12. For this purpose let $\bar{\mu}$ be an equilibrium state for a Hölder continuous function $F: \Sigma \to \mathbb{R}$. Let φ be a conjugating homeomorphism defined in some $\mathcal{O}_\varphi \subset \mathbf{\Sigma}$. Let $x \in \mathcal{O}_\varphi$, denote $y = \varphi(x)$, and let α, β be the delay functions, that is, $d(\Phi_{t+\alpha}(x), \Phi_t(y)) \to 0$ as $t \to -\infty$ and $d(\Phi_{t+\beta}(x), \Phi_t(y)) \to 0$ as $t \to \infty$. Then, by Proposition 11,

$$\frac{d\bar{\mu}(y)}{d\bar{\mu}(x)} = \frac{d\mu^{\mathrm{uu}}(y)\, d\mu^{\mathrm{ss}}(y)\, dt}{d\mu^{\mathrm{uu}}(x)\, d\mu^{\mathrm{ss}}(x)\, dt}$$

where

$$\begin{aligned}\frac{d\mu^{\mathrm{uu}}(y)}{d\mu^{\mathrm{uu}}(x)} &= \frac{d\mu^{\mathrm{uu}}(y)}{d\mu^{\mathrm{uu}}(\Phi_\beta(x))} \frac{d\mu^{\mathrm{uu}}(\Phi_\beta(x))}{d\mu^{\mathrm{uu}}(x)} \\ &= \exp(\omega_{\Phi_\beta(x), y} \circ \Phi_\beta(x) - \tau_\beta(x))\end{aligned}$$

since $\Phi_\beta(x)$ and y lie on the same strong stable leaf and

$$d\mu^{\mathrm{uu}}(\Phi_\beta(x)) = d(\Phi^*_{-\beta}\mu^{\mathrm{uu}})(x) = e^{-\tau_\beta(x)}\, d\mu^{\mathrm{uu}}(x)$$

On the other hand, we have

$$\begin{aligned}\frac{d\mu^{\mathrm{ss}}(y)}{d\mu^{\mathrm{ss}}(x)} &= \frac{d\mu^{\mathrm{ss}}(y)}{d\mu^{\mathrm{ss}}(\Phi_\alpha(x))} \cdot \frac{d\mu^{\mathrm{ss}}(\Phi_\alpha(x))}{d\mu^{\mathrm{ss}}(x)} \\ &= \exp[\omega'_{\Phi_\alpha(x), y} \circ \Phi_\alpha(x) + \tau_\alpha(x)]\end{aligned}$$

since $\Phi_\alpha(x)$ and y share the same strong unstable leaf.

Thus

$$\frac{d\bar{\mu}(y)}{d\bar{\mu}(x)} = \exp[\omega_{\Phi_\beta(x),\, y} \circ \Phi_\beta(x) + \omega'_{\Phi_\alpha(x),\, y} \circ \Phi_\alpha(x) - \tau_\beta(x) + \tau_\alpha(x)]$$

$$= \exp\left\{\int_{-\infty}^{0} (F \circ \Phi_{t+\alpha} \circ \varphi - F \circ \Phi_t)\, dt\right.$$

$$\left. + \int_\alpha^\beta [F \circ \Phi_t \circ \varphi - P(F)]\, dt + \int_0^\infty (F \circ \Phi_{t+\beta} \circ \varphi - F \circ \Phi_t)\, dt\right\}$$

where the second integral in the braces is equal to $\tau_\beta(x) - \tau_\alpha(x)$. This proves the first part of Proposition 12.

To prove the second statement of Proposition 12, let $\bar{\mu}$ be a Gibbs state on Σ for the family of multipliers $\{R_\varphi$: φ conjugating in $\Sigma\}$, where the multipliers R_φ are of the form given in $(*)$ for a Hölder continuous $F: \Sigma \to \mathbb{R}$. Since $\bar{\mu}$ is a Φ_t-invariant measure on the suspension Σ, it can be written as the product $\mu \times l/\mu(r)$, where l is the Lebesgue measure on $\mathbb{R}$ and μ is a shift-invariant probability measure on the shift space Σ. Our statement now follows immediately from Lemma 14(II) and Lemma 13 below. ∎

Lemma 13. (ref. 3, Proposition 3.1). Let $F: \Sigma \to \mathbb{R}$ be continuous and assume that $f(x) = \int_0^{r(x)} [F(x, s) - P(F)]\, ds$ is Hölder continuous of class $C_\theta(\Sigma)$. Then the equilibrium state $\bar{\mu}$ on Σ for F is given by $\bar{\mu} = \mu \times l/\mu(r)$, where μ is, on the shiftspace Σ, the equilibrium state for f.

Lemma 14. I. The probability measure μ on Σ associated to $\bar{\mu}$ on the suspension Σ is a Gibbs state for a family of discrete multipliers $\{r_\psi \in C_\theta(\Sigma)$: ψ conjugating$\}$.

II. If $\bar{\mu}$ on Σ is strongly Gibbs for F, then μ is an equilibrium state on Σ for the function $f(x) = \int_0^{r(x)} [F(x, t) - P(F)]\, dt$.

Proof. We show that a conjugating homeomorphism ψ on Σ gives rise to a conjugating homeomorphism φ on Σ and provides us thus with a family of multipliers r_ψ for the measure μ on Σ.

Let ψ be a conjugating homeomorphism defined on some $\mathcal{O}_\psi \subset \Sigma$, that is, for $z \in \mathcal{O}_\psi$ we have $(\psi z)_j = z_j$ for $|j| \geqslant N$ for some integer N. Then we can construct a conjugating homeomorphism φ on some $\mathcal{O}'_\varphi \subset \Sigma$ as follows: For points $(z, 0)$ in Σ we put $\varphi(z, 0) = (\psi(z), 0)$, $z \in \mathcal{O}_\psi$, and extend it to an open neighborhood by putting $\varphi(z, t) = \Psi_t(\psi(z), 0)$.

We have to show that φ is indeed conjugating, for which it is sufficient to show that the points $(z, 0)$ and $(\psi(z), 0)$ are conjugating. Let $t > 0$ be large; then we have $\Psi_t(z, 0) = (\sigma^{k(t)}(z), s) \in \mathbf{\Sigma}$, for some suitable integer $k(t)$ which satisfies $t = r^{k(t)}(z) + s$, where we used the abbreviation $r^k = r + r \circ \sigma + r \circ \sigma^2 + \cdots + r \circ \sigma^{k-1}$. Similarly, $\Psi_{t+\beta}(\psi(z), 0) = (\sigma^{k'(t+\beta)} \circ \psi(z), s') \in \mathbf{\Sigma}$, where $t + \beta = r^{k'(t+\beta)} \circ \psi(z) + s'$. The number β is now chosen so that $k'(t+\beta) = k(t)$ if

$$\delta < s, s' < \min(r \circ \sigma^{k(t)}(z), r \circ \sigma^{k'(t+\beta)} \circ \psi(z)) - \delta$$

for some small positive δ and all large enough t. This is achieved by putting

$$\beta = -\lim_{k \to \infty} [r^k \circ \psi(z) - r^k(z)] = -\sum_{k \geqslant 0} (r \circ \sigma^k \circ \psi - r \circ \sigma^k)(z)$$

The limit exists since z and $\psi(z)$ are conjugated points in Σ and $\beta(x)$ is Hölder continuous in x. In a similar fashion one shows that the other delay function α has to be chosen as

$$\alpha = \sum_{k \geqslant 1} (r \circ \sigma^{-k} \circ \psi - r \circ \sigma^{-k})(z)$$

This proves that a conjugating homeomorphism ψ on the subshift Σ can be extended to a conjugating homeomorphism φ on the suspended shift $\mathbf{\Sigma}$.

Now let ψ and φ be, as above, conjugating homeomorphisms in Σ and $\mathbf{\Sigma}$, respectively. Since

$$R_\varphi(x, t) = \frac{d\varphi^*\bar{\mu}}{d\bar{\mu}}(x, t) = \frac{d\mu(x')\, dt'}{d\mu(x)\, dt} = \frac{d\mu(x')}{d\mu(x)}$$

where $(x', t') = \varphi(x, t)$, it follows that $R_\varphi(x, t) = R_\varphi(x, 0)$ is independent of t and thus depends only on the shiftspace coordinate x. Thus we can define $r_\psi(z) = R_\varphi(z, 0)$ for $z \in \mathcal{O}_\psi \subset \Sigma$.

We will show that $\{r_\psi : \psi \text{ conjugating}\}$ is indeed a family of discrete multipliers on the shiftspace Σ. Put $\psi' = \sigma \circ \psi \circ \sigma^{-1}$ and let $\varphi' = \Psi_{r(\sigma^{-1}(z))} \circ \varphi \circ \Psi_{-r(\sigma^{-1}(z))}$ be the associated conjugating homeomorphism in $\mathbf{\Sigma}$. Then shift invariance of the multiplier r_ψ, $r_{\psi'} \circ \sigma = r_\psi$, follows immediately from the flow invariance of R_φ: $R_{\varphi'} \circ \Psi_{r(\sigma^{-1}(z))} = R_\varphi$. Now assume ψ', ψ'' are conjugating homeomorphisms in Σ such that the composition $\psi = \psi'' \circ \psi'$ is defined on some appropriate domain $\mathcal{O}_\psi \subset \Sigma$. Let φ' and φ'' be the associated conjugating maps in $\mathbf{\Sigma}$. For the composition we get for the points $(z, 0)$, $z \in \mathcal{O}_\psi$, that $\varphi(z, 0) = (\psi'' \circ \psi'(z), 0) = \varphi''(\psi'(z), 0) = \varphi'' \circ \varphi'(z, 0)$ and that indeed $\varphi = \varphi'' \circ \varphi'$, which implies the cocyle equation. This proves part I of the lemma.

To prove part II of the lemma, let $\bar{\mu}$ be strongly Gibbs for F, that is, the multipliers R_φ are given by $(*)$. Let ψ and φ be conjugating homeomorphisms as above on $\mathcal{O}_\psi \subset \Sigma$ and $\mathcal{O}_\varphi \subset \boldsymbol{\Sigma}$, respectively. We get that the discrete multipliers are

$$\begin{aligned} r_\psi(x) = \exp \sum_{k=-\infty}^{\infty} \bigg\{ & \int_{r^k(x)}^{r^{k+1}(x)} [F \circ \Psi_t(x, 0) - P(F)] \, dt \\ & - \int_{r^k(\psi(x))}^{r^{k+1}(\psi(x))} (F \circ \Psi_t(\psi(x), 0) - P(F)) \, dt \bigg\} \\ = \exp & \sum_{k=-\infty}^{\infty} [f \circ \sigma^k \circ \psi(x) - f \circ \sigma^k(x)] \end{aligned}$$

and thus satisfy (**) for the f given in the statement of the lemma. Thus μ is strongly Gibbs for f as in ref. 8, and therefore an equilibrium state for f. ∎

Next let us prove Theorem 6 for a suspended flow. We reformulate:

Proposition 15. Let $\{R_\varphi$: φ conjugating in $\boldsymbol{\Sigma}\}$ be a family of multipliers such that $R_\varphi(z, t_0) \in C_\theta(\Sigma)$ as functions of z for fixed t_0, where $\theta > h = h_{\text{top}}(\sigma)$; then there exists a Hölder continuous function $F: \boldsymbol{\Sigma} \to \mathbb{R}$ such that the multipliers R_φ are of the form given in (*). Moreover, $f(z) = \int_0^{r(z)} F(z, t) \, dt$ lies in the function space $\bigcup_{0 \leqslant \theta' < \theta - h} C_{\theta'/2}(\Sigma)$.

Proof. The goal is to find a Hölder continuous function $F(z, t)$ on the suspension $\boldsymbol{\Sigma}$ such that $\bar{\mu}$ is an equilibrium state for F. Let $\{R_\varphi$: φ conjugating in $\boldsymbol{\Sigma}\}$ be the given family of multipliers for the Gibbs state $\bar{\mu}$. Then, by Lemma 14, the associated measure μ on the subshift Σ is Gibbs with respect to the family of discrete multipliers $\{r_\psi \in C_\theta(\Sigma)$: ψ conjugating$\}$. By Proposition 8 there exists a Hölder continuous function f in $\bigcup_{0 < \theta' < \theta - h} C_{\theta'/2}(\Sigma)$, where $h = h_{\text{top}}(\sigma) < \theta$, such that the discrete multipliers r_ψ are given by $r_\psi = \exp \sum_k (f \circ \sigma^k \circ \psi - f \circ \sigma^k)$ (ψ conjugating) and so that μ is the equilibrium state for f.

Let us assume that $p(f) = 0$ [otherwise replace f by the function $f - p(f)$, which has the same equilibrium state μ and the same discrete multipliers r_ψ as f] and define the potential function F on the suspension $\boldsymbol{\Sigma}$ as follows. Put $\delta = \min_{z \in \Sigma} r(z)/2$ and define ($\delta > 0$):

(i) $F(z, t) = f(z)(t/\delta^2)$ if $0 \leqslant t \leqslant \delta$

(ii) $F(z, t) = [f(z)/\delta](2 - t/\delta)$ if $\delta \leqslant t \leqslant 2\delta$

(iii) $F(z, t) = 0$ if $2\delta \leqslant t \leqslant r(z)$

Obviously $F(z, t)$ is Hölder continuous, has pressure zero since $P(F) = p(f) = 0$, and yields $f(z) = \int_0^{r(z)} F(z, t) \, dt$. This concludes the proof of Proposition 15. ∎

7. MARKOV PARTITIONS AND THE PROOF OF THEOREMS 5 AND 6

In this last section we use standard arguments for Markov partitions to generalize our results of the last section from suspended flows over subshifts of finite type to arbitrary Axiom A flows. Markov partitions for Axiom A flows are constructed in much the same way as for diffeomorphisms. We will summarize the procedure and refer for details to ref. 2. One can chose finitely many arbitrarily small pieces of hyperplanes H_j transverse to the flow and define the Poincaré map P: $\bigcup_j H_j \to \bigcup_j H_j$ by $P(x) = \Phi_{\hat{r}(x)}(x)$, where $\hat{r}(x)$, the *ceiling function*, measures the time it takes for a point x on some H_j to flow up to the next cross section, i.e., for $x \in \bigcup_j H_j$ we define $\hat{r}(x) = \min\{t > 0\colon\ \Phi(x) \in \bigcup_j H_j\}$. The sections are chosen to satisfy the so-called Markov conditions, which are:

(i) $\operatorname{clos}(\operatorname{int}(H_j)) = H_j$ for all j

(ii) If $x \in H_k$, $P(x) \in H_j$, then

$$(\Phi_{[0,2\delta]}(W^u_\varepsilon(x) \cap H_k)) \cap H_j = W^u_\varepsilon(P(x)) \cap H_j$$

$$(\Phi_{[-2\delta,0]}(W^s_\varepsilon(P(x)) \cap H_j)) \cap H_k = W^s_\varepsilon(x) \cap H_k$$

where ε is the size of the pieces of hyperplanes, i.e., $\operatorname{diam}(H_j) \leqslant \varepsilon$ for all j, and $\delta = \|r\|_\infty$. One defines a transition matrix A by putting $A_{i,j} = 1$ if $P(H_i) \cap H_j \neq \emptyset$ and $A_{i,j} = 0$ otherwise. The subshift Σ now consists of all points $z \in \prod_{-\infty < i < \infty} \{1,..., n\}$, $n = \# H_j$, which satisfy the transition condition $A_{x_i, x_{i+1}} = 1$ for all i. The map $\pi\colon \Sigma \to \bigcup_j H_j$ is defined by $\pi(z) = \bigcap_{-\infty < i < \infty} P^{-i}(H_{z_i})$, where the infinite intersection consists of exactly one point if ε is small enough. If we denote by σ the shift transformation on Σ, $\sigma(z)_i = z_{i+1}$, then $\pi \circ \varphi = P \circ \pi$. The function $r = \pi_* \hat{r}$ is a Hölder continuous function in Σ, which, suspended over the shiftspace Σ, defines the suspension $\mathbf{\Sigma} = \{(x, t) \in \Sigma \times \mathbb{R}\colon\ 0 \leqslant t \leqslant r(x)\}$, with the identification $(x, r(x)) = (\sigma(x), 0)$, on which we define as above the suspended flow Ψ_t always keeping in mind that we identify points $(x, r(x))$ with $(\sigma(x), 0)$. The map $\pi\colon \Sigma \to \bigcup_j H_j$ is easily extended to a map $\Pi\colon \mathbf{\Sigma} \to \Omega$ such that $\Pi \circ \Psi_t = \Phi_t \circ \Pi$.

A conjugating homeomorphism φ in the suspension $\mathbf{\Sigma}$ gives rise to a conjugating homeomorphism $\hat{\varphi}$ on Ω such that $\hat{\varphi} \circ \Pi = \Pi \circ \varphi$. Moreover, if x is a point in $\mathbf{\Sigma}$ (or Ω), then the set of points conjugated to it is dense in $\mathbf{\Sigma}$ (respectively Ω). In fact, Proposition 12 describes how a certain type of conjugating map in $\mathbf{\Sigma}$ can be constructed from conjugating maps in the shift Σ.

Let $\hat{\mu}$ be a Gibbs state for a family of multipliers $\{\hat{R}_{\hat{\varphi}}\colon\ \varphi$ conjugating in $\Omega\}$. The map Π is one-to-one almost everywhere for ergodic measures that are positive on open sets. We consequently have to show that the

$\hat{\mu}$-measure of the set of which Π is not one-to-one is zero. Let $\partial H = \bigcup_k \partial H_k$ be the collective boundary of the hyperplanes H_k and put $K = \bigcup_j T^j(\partial H)$. Then K is on the cross section $\bigcup_k H_k$ the set of points on which π is not one-to-one. Thus $\hat{K} = \Phi_{[0,\hat{r}]}(K) = \{\Phi(y);\ 0 \leqslant t \leqslant \hat{r}(y),\ y \in K\}$ is where Π fails to be one-to-one. For the following lemma see also ref. 8.

Lemma 16. If $\hat{\mu}$ is a Gibbs state on Ω, then $\hat{\mu}(\hat{K}) = 0$.

Proof. Assume $\hat{\mu}(\hat{K}) \neq 0$. Put $\partial^+ H$ for the forward boundary of the Markov partition and $\partial^- H$ for the backward boundary. Then we have $\partial H = \partial^+ H \cup \partial^- H$, $P(\partial^+ H) \subset \partial^+ H$, and $P^{-1}(\partial^- H) \subset \partial^- H$. Put $H^{\pm}$ for the compact and P-invariant sets $\bigcap_{j \geqslant 0} P^{\pm j}(\partial^{\pm} H)$, let $K^{\pm} = \Phi_{[0,\hat{r}]}(H^{\pm})$, and note that, since $\hat{\mu}$ is flow invariant, either $\hat{\mu}(K^+) \neq 0$ or $\hat{\mu}(K^-) \neq 0$ (or both). Assume $\hat{\mu}(K^+) \neq 0$ and let $\varepsilon > 0$. Then $\hat{\mu}(K^+ \cap B_\varepsilon(x)) \neq 0$ for some point x in K^+, where $B_\varepsilon(x)$ is the ball of radius ε and center x. Choose $y \notin B_\varepsilon(x)$ such that there exists a conjugating homeomorphism $\hat{\varphi}$ from $B_\varepsilon(x)$ into a neighborhood $\mathcal{O}$ of y with $\mathcal{O} \cap B_\varepsilon(x) = \varnothing$ and satisfying $\operatorname{dist}(\Phi_\beta \circ \hat{\varphi}(B_\varepsilon(x)), K^+) = \delta > 0$ (β is a delay function for φ), where $\Phi_\beta \circ \hat{\varphi}(B_\varepsilon(x)) = \{\Phi_{\beta(y)} \circ \hat{\varphi}(y)\colon y \in B_\varepsilon(x)\}$ (for ε sufficiently small). Now let $T > 0$ be such that

$$\sup_{y \in \Phi_\beta \circ \hat{\varphi}(B_\varepsilon(x))} \operatorname{dist}(\Phi_T(y), K^+) < \delta$$

Thus we have $\Phi_T(D)$ and D are disjoint, where $D = \Phi_\beta \circ \hat{\varphi}(K^+ \cap B_\varepsilon(x))$, and also $\Phi_{kT}(D) \cap D = \varnothing$ for all $k = 1, 2, \ldots$. Thus the sets $\{\Phi_{kT}(D)\colon k = 0, 1, 2, \ldots\}$ are pairwise disjoint, and since $\hat{\mu}(D) > 0$, this contradicts finiteness of the measure $\hat{\mu}$. ∎

Proof of Theorem 5. To prove the first part of general Axiom A flows let $\hat{F}$ be a Hölder continuous function on Ω with Hölder exponent $\eta > 0$ and $\hat{\mu}$ its equilibrium state. By ref. 9, Theorem 7.6, $\hat{\mu} = \Pi^* \bar{\mu}$, where $\bar{\mu}$ is the equilibrium state for the lifted function $F = \Pi_* \hat{F}$ and is, by Proposition 12, also strongly Gibbs for F. This proves the first part of Theorem 5. Note that $f(z) = \int_0^{r(z)} F(z, t)\, dt$ Hölder continuous of class $C_\theta(\Sigma)$, where $\theta = \eta\lambda/2$.

By Lemma 16, a Gibbs state $\hat{\mu}$ for the family of multipliers $\{\hat{R}_{\hat{\varphi}}\colon \hat{\varphi}$ conjugating in $\Omega\}$ lifts to a Gibbs state $\bar{\mu} = \Pi^{*-1}\hat{\mu}$ [where $\bar{\mu}(\Pi^{-1}(\hat{K})) = 0$] on Σ with multipliers $R_\varphi = \Pi_* \hat{R}_{\Pi \circ \varphi}$, where φ are conjugating homeomorphisms in Σ. Now assume that $\hat{\mu}$ is Gibbs in the strong sense for some Hölder continuous function $\hat{F}$; then $\bar{\mu}$ is strongly Gibbs for $F = \Pi_* F$ and by Proposition 12(ii) also the equilibrium state for F. This proves Theorem 5(ii). ∎

Proof of Theorem 6. If we assume that the multipliers $\hat{R}_{\Pi \circ \varphi}$ are Hölder continuous to the exponent η, then their lifts $R_\varphi(x, t) = \Pi_* \hat{R}_{\Pi \circ \varphi}$ will for fixed t, as functions of x, be of class $C_\theta(\Sigma)$ with $\theta = \eta\lambda/2$. By Proposition 15 there exists a Hölder continuous function F on the suspension $\boldsymbol{\Sigma}$ which has the equilibrium state $\bar{\mu}$, provided $\theta > h_{\text{top}}(\sigma)$. It therefore remains to show that θ is indeed larger than the topological entropy of the subshift Σ.

A flow-invariant measure $\hat{\rho}$ on $\boldsymbol{\Sigma}$ is given by $\hat{\rho} = \rho \times l/\rho(r)$, where ρ is a shift invariant measure on Σ and l is Lebesgue measure on $\mathbb{R}$. By Abramov's formula,[1] $h(\rho) = H(\hat{\rho})\, \rho(r)$, where $H(\hat{\rho})$ is the metric entropy of $\hat{\rho}$ and $h(\rho)$ is the metric entropy of ρ, and by the variational principle we have

$$h_{\text{top}}(\sigma) = \sup_{\rho} h(\rho) \leqslant \sup_{\hat{\rho}} H(\hat{\rho})\, \|r\|_\infty = H_{\text{top}}(\Phi_t)\, \|r\|_\infty$$

Moreover, since $\theta = \frac{1}{2}\lambda\eta \cdot \inf r$, we get

$$h_{\text{top}}(\sigma) \leqslant H_{\text{top}}(\Phi_t) \cdot \sup r = \frac{6H_{\text{top}}(\Phi_t)}{\lambda} \frac{\lambda}{2} \inf r < \eta \frac{\lambda}{2} \inf r = \theta$$

Here we made use of the fact that the Markov partition can be chosen so that the condition $\sup r \leqslant 3 \cdot \inf r$ is satisfied. Thus we have $\theta > h_{\text{top}}(\sigma)$, since η is by assumption greater than $6H_{\text{top}}(\Phi_t)/\lambda$. ∎

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